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ESTIMATING LEARNING CURVES FROM AGGREGATE MONTHLY DATA*

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In this paper the problems of using aggregate monthly data to estimate learning curves are investigated. Here, aggregate monthly data on labor hours are assumed to contain some of both fixed and variable labor hours. They are also assumed to be influenced by fluctuating quantities of work in process. A distributed lag model is developed to deal with these two characteristics of aggregate monthly data. The model is generalized to permit production rate to influence labor productivity. This generalized model is then estimated and compared to a cumulative average learning curve in analyzing the impact of a production break. A set of production data which arose from a government contract claim is used for this purpose.
(PRODUCTION/SCHEDULING—WORK STUDIES; FORECASTING—APPLICATIONS; LABOR)

1. Introduction

Yelle's (1979) survey lists 93 references to recent applications of the learning curve. Rather clearly, the learning curve concept is thought to be important as both a descriptive model and a decision making tool in a wide variety of settings.

Frequently, learning curves are estimated from the direct labor hours expended in the production of each successive unit of a product. In some cases, where unit data are unavailable, direct labor hours expended on each successive lot of units and the number of units in each lot are used instead. See Asher (1956), Berend (1977), and Conway and Schultz (1959) for a discussion of learning curve estimation in these environments.

In a data collection system designed for estimating learning curves, labor hours would be collected either by unit produced or by lot. But sometimes the data collection system which must be used is not designed for learning curve estimation. Frequently this is the case when only historical data are available for learning curve estimation. This situation might also characterize ongoing production efforts where the cost of collecting unit cost data is prohibitive.

In this paper learning curve estimation is investigated in an environment where labor hours and number of units produced are collected only for some unit of time, say a month. That is, the available data do not directly relate labor hours and a specific group of units to each other, but only to the month of observation. Others have estimated learning curves in this data regime. Baloff (1971) used monthly and quarterly accounting data to estimate learning curves for a musical instrument, for apparel manufacture and for automobile assembly. Likewise the data provided by the Boston Consulting Group (1970) are annual data on various equipment.

The monthly data environment may mask two phenomena which, if not adequately modeled, can severely bias the estimated learning curve. First, the production period for a unit may be much longer than one month. Thus the units shipped in one month result from labor in, perhaps, several previous months. This results in substantial work in process at the end of each month. A second phenomenon may also be masked by this data problem. Labor hours are not reported by unit. So it is possible that some of

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the labor reported as direct labor hours may, in fact, be indirect labor hours. That is, an accounting system that does not track labor by unit may fail to discriminate direct labor from indirect labor as accurately as one that does. But indirect labor might not be subject to learning at all.

Estimating learning curves without adequately modeling the data generating situation can lead to serious errors—errors that can completely distort a decision maker's perception of the production situation and lead to quite inappropriate decisions. In the next section a numerical example is provided which illustrates some spurious learning effects which are estimated from aggregate monthly data. A distributed lag model to deal with aggregate monthly data is developed in §3. It is estimated and applied in §4.

2. An Illustrative Example

The consequences of estimating a cumulative average learning curve directly from monthly data without modeling work in process or considering misclassified labor hours are demonstrated by the following numerical example. Conway and Schultz (1959) anticipate some of these consequences, "the cumulative average formulation of the progress function is overused—primarily because the averaging process has tremendous power to smooth the data and enhance the appearance if not the substance of the curve."

There is no learning in the production of widgets. Each widget requires three hours of labor for its production, one in each of the three months preceding shipment. In addition, to operate the widget plant, one labor hour is required for each month in which any widget production is taking place. This hour is consumed by jobs such as supervision, replacing bench stock, trips to the tool crib, coffee breaks, and other once a day activities, activities that would ordinarily be classified as indirect labor. However, since the accounting system does not track labor by units, this is referred to as direct supervision and is included in direct labor hours.

The aggregate monthly data on production activity in the widget factory over a 15-month period are listed in Table 1. Seven shipments of two widgets each are made in the months March through September; no widgets are shipped from October through January; one widget is shipped in February and another in March. The corresponding data on aggregate monthly labor hours reflect the one hour per month

TABLE 1

Aggregate Monthly Data on the Widget Factory

Month	Units Shipped	Monthly Labor Hours	Cumulative Units Shipped	Cumulative Labor Hrs.	Cumulative Avg/Labor/Hrs
Jan	0	3	0	3	—
Feb	0	5	0	8	—
Mar	2	7	2	15	7.5
Apr	2	7	4	22	5.5
May	2	7	6	29	4.83
Jun	2	7	8	36	4.5
Jul	2	7	10	43	4.3
Aug	2	5	12	48	4.0
Sep	2	3	14	51	3.64
Oct	0	0	14	51	—
Nov	0	0	14	51	—
Dec	0	2	14	53	—
Jan	0	3	14	56	—
Feb	1	3	15	59	3.93
Mar	1	2	16	61	3.81

of direct supervision, and the labor used to produce work in process at the end of the month.

It seems clear that a unit learning curve does not apply to these data. In the first two months labor hours are expended but no units have yet been shipped. Furthermore, the entire period from October to January is not likely to fit a unit curve. On the other hand, if the cumulative average labor hours are calculated, the situation appears to be more compatible with learning. Then the set of data seems to follow the usual learning curve pattern from March through September. In October an apparent two-month break in production occurs. If the log of cumulative average labor hours from March through September is regressed on the corresponding cumulative units shipped, a cumulative average learning curve seems to explain the data very well. The relation is estimated as

$$C_j = 9.25 Z_j^{-0.347} \tag{1}$$

where C_j is cumulative average direct labor hours though month j and Z_j is cumulative widgets shipped through month j.

The exponent (-0.347) indicates a 78.6% learning rate. The exponent and the intercept, the first unit cost, appear to be statistically significant at the 5% level. The value of R^2 is 0.98 indicating a close fit to the data. Thus, the effect of ignoring the problems of work in process and direct supervision is to estimate a 78.6% learning curve from data that were generated with no learning at all. The data and the estimated learning curve are plotted in Figure 1.

To continue the illustration a bit further, the production situation after October is compared to the estimated learning curve. From Table 1 it is clear that 10 labor hours were incurred in the production of the last two widgets. But, extrapolating along the estimated learning curve yields 3.56 cumulative average labor hours required to produce 16 widgets. Therefore, total labor predicted for 16 widgets is 56.6 hours and the additional labor predicted for the last two widgets is 4.7 hours. Following Anderlohr (1969) one might be tempted to conclude that a break in production of two months resulted in loss of learning of 5.3 hours, or 53% of the hours required to produce the last two widgets. The data in Figure 1 are subject to other interpretations as well. If there were no apparent break in production, two data points above the regression lines at the end of the production run might illustrate "toe up." Likewise, had the run stopped in October, the data would seem to exhibit "toe down." See Russell (1968) for more discussion of these phenomena.

In fact, the increased unit cost of the February and March deliveries is due to both

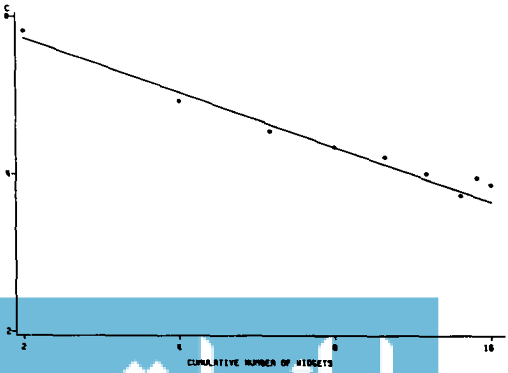

FIGURE 1. Estimated Cumulative Average Learning Curve

the production break and the reduced production rate, but it has nothing at all to do with learning. The unit cost of widgets increases merely because four hours of direct supervision (December though March) are spread over only two units.

This example emphasizes the bias that can be introduced into analyses by ignoring the data problems of work in process and misclassified labor hours. Mistreating the data by ignoring these problems introduces a persistent bias towards learning into the analysis. This bias can be critical when evaluating the effect of a production break.

Of course, the data for this example were generated with no learning and both data problems present. If the labor content of the work in process each month were not known, it would not be clear if learning were present in the data or not. If the data were more realistic, with random errors and varying quantities shipped, it would be even less clear whether learning applied to data and, if so, how to estimate a learning curve. A model to deal with this and more general situations is developed in the next section.

3. The Lagged Model

When data are available by "Lots" of units, a usual form of the learning curve is

$$L_i / U_i = \alpha (X_i)^{\beta} \qquad \text{where} \tag{2}$$

L_i = labor hours expended on the ith lot,
U_i = number of units in the ith lot,
X_i = cumulative number of units produced through the midpoint of the ith lot.

Equation (2) is a unit learning curve which relates average lot labor hours to cumulative production. α and β are parameters to be estimated in equation (2). Even though (2) is a different model from (1), α and 2^{β} are often referred to as the "first unit cost" and the "slope" of the unit learning curve just as the intercept and the slope of the cumulative average learning curve were. For lot learning curves this interpretation is appropriate only if X_i is the "true lot midpoint." (See Berend 1977 for more details on this.)

From (2) the labor hours required for each lot is given by

$$L_i = \alpha X_i^{\beta} U_i . \tag{3}$$

The aggregate monthly data used in the next section are not collected by lots of units. Nevertheless, to use this model, we assign all the units that are shipped in the same month to the same lot. This designation permits the subscript i to stand for both the ith lot and the ith month of shipment. So U_i, the size of the ith lot, is also the number of units shipped in month i. Since U_i measures output per unit of time, it can be regarded as a measure of production rate.

The model at (3) can now be generalized to permit production rate to affect unit labor hours. This generalization is not new. Asher (1956, p. 86) discusses the impact of production rate on learning curves and Conway and Shultz (1959, p. 42) refer to a learning curve that is sensitive to production rate as the "generalized progress function." The generalization is included in studies by Alchian (1959), Hirshleifer (1965), Oi (1967), and Womer (1979, 1981). Successful empirical work on generalized learning curves is rather scarce, but Smith (1976) and Washburn (1972) have made recent contributions.

A convenient generalization of the learning curve at (3) is

$$L_i = \alpha X_i^{\beta} U_i^{\gamma} \qquad \text{where} \tag{4}$$

γ is a parameter describing returns to scale in the production process.

This model is a special case of the model in Womer (1979), but here U_i is assumed to be an exogenous variable, not controlled by the producer. In (4), α can be thought of as the first unit cost corresponding to a production rate of one unit per month.

Since the direct labor hours expended on the units shipped in month i (lot i) are not observable, equation (4) cannot be directly applied to the data. The direct labor hours that are expended in month j, D_j, are expended on perhaps several lots of monthly shipments, say $j, j+1, j+2, \ldots, j+k$, where k is the number of months required to produce a unit. To make use of equation (4) we assume that the direct labor hours expended in month j on units to be shipped in month i (lot i) are some fraction of all the direct labor hours expended on the units in lot i. Furthermore, we assume that the fraction, δ_{i-j}, depends only on the difference $(i-j)$, the number of months until shipment. This assumption requires that production is organized as a fixed sequence of events which does not change as learning takes place and as production rate changes. Furthermore, it requires that changes in labor requirements are distributed proportionately across the events. If L_{ij} represents the labor hours expended on lot i in a preceding month j then this assumption requires

$$L_{ij} = \delta_{(i-j)}(L_i). \tag{5}$$

Letting $h = i - j$ it is also assumed that

$$\delta_h = 0 \quad \text{for} \quad h > k \quad \text{and} \quad \sum_{h=0}^{k} \delta_h = 1. \tag{6}$$

That is, no labor is expended on the units to be shipped in months more than k months in advance of the month of shipment, and the sum of the labor expended on units shipped in month i over the k ·: 1 months ending with month i is the total labor expend on lot i.

Substituting from (4) into (5) yields:

$$L_{ij} = \delta_{(i-j)}\alpha X_i^{\beta} U_i^{\gamma} \qquad (j = i-k, i-k-1, \ldots, i). \tag{7}$$

Equation (7) relates some of the labor hours in month j, L_{ij}, to units which will not be completed until month i. Even though these units have not yet been completed, they measure experience on the production events that are relevant to L_{ij}. Summing (7) over the $k+1$ months ending with month i yields the labor hours required for lot i:

$$L_i = \sum_{j=i-k}^{i} L_{ij} = \sum_{j=i-k}^{i} \delta_{(i-j)}\alpha X_i^{\beta} U_i^{\gamma} = \alpha X_i^{\beta} U_i^{\gamma}, \tag{8}$$

as required by (6).

Under these assumptions, the learning curve of (4) can be thought of as the sum of the $k+1$ individual learning curves at (7). The representative learning curve applies to the activity that precedes shipment by $i-j$ months. This is similar to the disaggregation-aggregation approach described by Conway and Schultz (1959), but here γ and β are the same for each of the curves.

Summing (7) over all the lots (i) that are in process during month j yields:

$$D_j = \sum_{i=j}^{j+k} L_{ij} = \sum_{i=j}^{j+k} \delta_{(i-j)}\alpha X_i^{\beta} U_i^{\gamma} \tag{9}$$

where D_j is total hours expended in month j on all lots.

The labor hours expended in month j, M_j, are given by

$$M_j = L_0 + D_j \tag{10}$$

where L_0 represents labor hours expended that are elements of fixed cost, unaffected by either production rate or learning.

In equation (9) α and the set of δ's are not all estimable. Letting $h = i - j$, the model is reparameterized by setting $d_h = \delta_{(i-j)}\alpha$. Estimates of the δ's and α may then be derived using (6) and (9). Substituting (9) into (10) yields

$$M_j = L_0 + \sum_{h=0}^{k} d_h X_{j+h}^{\beta} U_{j+h}^{\gamma}. \tag{11}$$

Equation (11) is a finite distributed lag model. Given the assumptions at (5) and (6), it is consistent with a generalized learning curve. In addition, the model at (11) permits some of the labor hours incurred each month to be elements of fixed cost, independent of production rate or learning.

The lagged model can be estimated using nonlinear least squares. Comparing the model to the data in Table 1 we see that if $k = 2$, $L_0 = 1$, $d_h = 1$, $\beta = 0$ and $\gamma = 1$ the lagged model fits the widget data exactly. In the next section the lagged model is applied to some production data which is characterized by varying monthly shipments.

4. The Lagged Model in Use

This application grew out of a claim on government contract. The contract was for some 540 made-to-order units of an electric-mechanical product. The production data examined arose from assembling and wiring complex electrical and mechanical components, testing the product and adjusting it prior to shipment.

The production effort was manually paced and involved mostly hand work. This portion of the manufacturing effort averaged over 1000 manhours per unit for the six-year life of the contract. The contractor's claim was based on the analysis of monthly direct labor hours and measures of monthly output over a 64-month period. The series on monthly output and on monthly labor hours are graphed in Figure 2.

Figure 2 shows the widely fluctuating levels of output rate and direct labor hours over time. The contractor argued that Figure 2 shows a production break in month 47. Month 48 began a six-month period during which no units were shipped. Likewise, for five months starting in month 47 an average of only 441.3 direct labor hours per month, only 4% of the average monthly labor hours, were expended on the program. The contractor argued that the production break resulted in a loss of learning. He asked to be compensated for the cost of this lost learning.

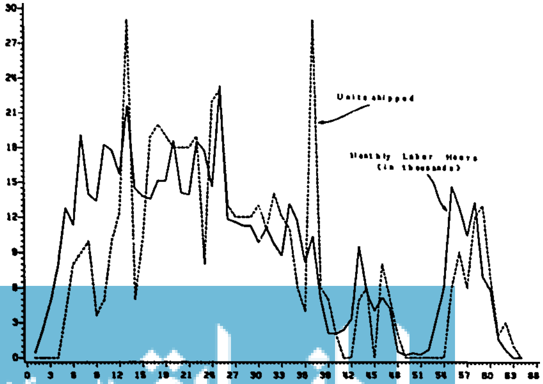

FIGURE 2. Monthly Labor Hours and Output.

No information was available which related the number of units produced to the labor hours required for their production. The direct labor hours were reported by month, not by unit. The hours reported included both the time of supervisors and "hands on" production time, but one can't tell how much of each is included. Some labor hours were probably expended on units over at least a four-month period prior to their shipment. This is inferred from contract provisions which call for shipments no more than 150 days after receipt of orders and from the contractor's original production plan which scheduled production activity up to 140 days in advance of shipment.

Because of these data problems, the lagged model at (12) was applied to the data. The initial value of k in the lagged model was taken to be four. The parameters of (11) were estimated using Marquardt's (1963) method of nonlinear least squares as implemented in Robinson (1977).

Preliminary analysis showed that it was not necessary to make use of units shipped four or more months in the future to explain monthly labor hours (a three-month lag structure was always sufficient to explain the data). This reflects the fact that only a small portion of the work on a unit was planned to start as much as 140 days prior to shipment.

The statistical results of estimating the lagged model on the first 47 months of data are displayed in Table 2. The first column of Table 2 reports results for the model at (11) estimated without restriction. While the unrestricted model provides a good fit to the data, the intercept, L_0, is estimated to be negative. L_0 is not significantly different from zero given the rest of the model, however.

The restriction that $L_0 = 0$ was then imposed on the model. This restriction implies that all the labor hours each month are variable and subject to learning; it resulted in a slight increase in the RSS. All of the coefficients except d_1 are still estimated to be significant at approximately the 5 percent level. In the case of d_1 the estimate to standard deviation ratio is 1.7, indicating a high level of significance for d_1 as well.

Two other regressions are reported in Table 2. In both these cases γ (the coefficient of production rate) is restricted to be 1.0. With this restriction, permitting L_0 to be greater than zero reduces the residual sum of squares by approximately 16 percent. Nevertheless, the model with γ unrestricted has substantially smaller residual sum of

TABLE 2

Statistical Results for the Lagged Model

Coefficients	Model Restrictions			
	None	$L_0 = 0$, γ Unrestricted	L_0 Unrestricted, $\gamma = 1.0$	$L_0 = 0$, $\gamma = 1.0$
L_0	−0.44	0.00*F*	2.29*	0.0*F*
d_0	3.18*	2.95*	1.06*	1.13*
d_1	0.87	0.78	0.34*	0.43*
d_2	1.22*	1.14*	0.54*	0.63*
d_3	1.88*	1.70*	0.61*	0.73*
β	−0.14*	−0.14*	−0.21*	−0.20*
γ	0.55*	0.59*	1.0*F*	1.0*F*
Statistics**				
RSS	253.0	253.6	290.8	347.9
R^2	0.83	0.83	0.80	0.76
df	40	41	41	42

*This estimate is significantly different from zero at the 5% level based on the linear approximation to the 95% confidence intervals calculated by the SPSS Nonlinear Subprogram (Robinson 1977).

**RSS is the residual sum of squares; R^2 is the proportion of variation in monthly labor hours explained by the model and df is the degrees of freedom for the model.

squares than either of the models with γ restricted to 1.0. Furthermore, the estimated value of γ in column two, 0.59, is significantly different from 1.0, thereby indicating a nonlinear relation between production rate and monthly labor hours and economies of increased production rate.

The lagged model was developed to overcome the data problems created by work in process and misclassified labor hours discussed above; but it has an additional advantage. If these data problems are not present, then the lagged model reduces to an ordinary unit learning curve. In our case however, γ and d_0 through d_3 are estimated to be significantly different from zero, indicating that the more complex lagged model is necessary to adequately explain the data.

The model with L_0 restricted to zero (no fixed labor hours) and γ unrestricted is therefore recommended as an adequate description of monthly labor hours. Using the coefficients in the second column of Table 2 the lagged model is

$$M_j = 2.95\, U_j^{0.59} X_j^{-0.14} + 0.78\, U_{j+1}^{0.59} X_{j+1}^{-0.14} + 1.14\, U_{j+2}^{0.59} X_{j+2}^{-0.14} + 1.70\, U_{j+3}^{0.59} X_{j+3}^{-0.14}. \quad (12)$$

This corresponds to a 91% unit learning curve with a first unit cost of 6.57 at a one unit per month production rate.

The contractor used the data without regard to work in process, production rate and misclassified labor. Like the numerical example in §2, the contractor estimated a cumulative average learning curve by taking logarithms of the cumulative average data.

The contractor's learning curve, estimated from the first 47 months of data, is

$$C_j = 8.65 Z_j^{-0.34}. \quad (13)$$

Z_j is the cumulative number of units shipped through month j, and $C_j = \sum_{i=1}^{j} M_i / Z_j$ is the cumulative average labor hours incurred through month j. The exponent of -0.34 corresponds to a 79% learning rate.

The coefficients of the contractor's learning curve are statistically significant at the 5 percent level, and R^2 is 0.99 indicating a very good fit to the logarithm of the cumulative average data. The Durbin–Watson statistic for (13) is 0.49. This indicates that the residuals of the model are subject to positive first-order autocorrelation. The Durbin–Watson statistic for the lagged model is 1.81 which indicates no such problem.

The contractor's learning curve (13) is estimated after a logarithmic transformation of the cumulative average data while the lagged model (12) is estimated without transforming the monthly data. Therefore, the statistics of the two models are not comparable. The learning curve (13) and the lagged model (12) can be compared by their ability to explain cumulative labor hours, however.

The differences between the cumulative hours estimated by the contractor's learning curve and the cumulative hours provided as data are plotted as a dashed line in Figure 3. The differences estimated by the lagged model are plotted as a solid line in Figure 3. For reference, the horizontal line shows the plot for a perfect fit, that is, a zero difference. Points below the horizontal line indicate estimates lower than actual cumulative hours. In 35 of the 47 months the lagged model differences are smaller than the learning curve's. Of course, the lagged model should fit the data better, it uses 5 more parameters than the learning curve.

The learning curve predicts that no hours will be required in months when there are no shipments. As a result, the learning curve is at a disadvantage in this comparison. To compensate for this the first five months are deleted from the comparison. Calculating the mean squared error (MSE) over the remaining 42 months gives a MSE

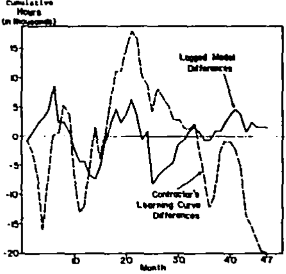

FIGURE 3. Differences between Cumulative Labor Hours and Predicted Cumulative Labor Hours by Month Prior to the Production Break.

for the learning curve of 86.7 and a MSE for the lagged model of 15.7. It seems clear that the lagged model does a better job of fitting cumulative labor hours over this period.

In Figure 4 the plots of differences are continued beyond month 47. As expected, the learning curve fails to adequately describe the data in this period. At the end of the contract the difference between the cumulative labor hours predicted by the learning curve and the actual hours incurred is about 52,000 labor hours. Following Adler and Nanda (1974) and Anderlohr (1969), the contractor argued that this was due to loss of learning. The government argued that the learning curve's misspecification and its inability to model work in process was at fault.

During the same period the lagged model continues to predict cumulative labor hours relatively well, ending with an under prediction of some 5,000 hours. The lagged model also explains the monthly data after month 47 very well. The MSE for the lagged model on the monthly data (not the cumulative data) for the first 47 months is 5.40, but for the period after the production break, the MSE is only 4.19. That is, the lagged model, estimated from data through month 47, fits the data after month 47 even better than the data from which it is estimated.

More formally, estimating the lagged model using only the observations after month 47 yields a residual sum of squares, RSS_a, of 25.47. Pooling all 64 observations yields RSS_p of 319.11. Letting the residual sum of squares for (12) be RSS_b, an approximate

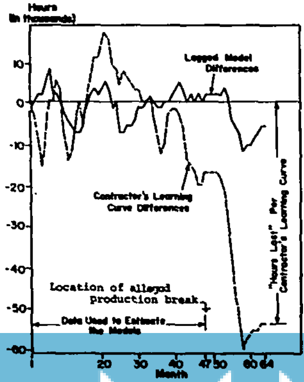

FIGURE 4. Differences between Cumulative Labor Hours and Predicted Cumulative Labor Hours by Month.

F statistic, (see Theil 1971, p. 147) can be calculated as:

$$F_{(6,52)} \simeq \frac{(n-2k)}{k} \frac{RSS_p - RSS_b - RSS_a}{RSS_b + RSS_a}$$

$$\simeq \frac{(64-2(6))}{6}\left[\frac{319.1 - 253.6 - 25.5}{277.1}\right] = 1.316. \quad (14)$$

This is not significant at the 5% level. Therefore, the hypothesis that a structural change in the relation occurred at month 47 is rejected.

We conclude that the lagged model is the more complete specification of the situation before the alleged production break, that the lagged model adequately describes the situation after the alleged break in production, and that the alleged break in production introduced no significant change in the situation.

The lagged model does not refute the contractor's claim that unit labor hours were higher after month 47; of course, they were higher. The period from month 48 to the end of the contract is characterized by a somewhat lower production rate than the previous 47 months. The lagged model permits this lower production rate to impact monthly labor hours both directly and through its impact on work in process. It seems clear that the significant changes in production that took place after month 47 are explained better by changes in work in process and reductions in production rate than by a loss of learning. If the customer were to be blamed for the reduced production rates and if the contract permitted compensation for these reductions, then the lagged model would be an appropriate basis for estimating the costs of the reduced rates.

5. Conclusions

In the paper, the problems of using monthly data to estimate learning curves have been discussed and illustrated. A new tractable procedure for estimating generalized learning curves from monthly data was developed. This finite distributed lag procedure was applied to a data set, and the results were compared to those from a cumulative average learning curve estimated from the same data. At least for this data, it was found that the lagged model provided the more appropriate specification of the situation. The lagged model was then applied to the problem of estimating the impact of claimed break in production. Here it was found that the changes in the monthly labor hours after the production break were nicely explained by the lagged model estimated prior to the claimed break. The impact of the claimed break was therefore limited to its impact on work in process and production rate, and no loss of learning was noticeable. This result points out a difficulty in applying previous work on the impacts of production breaks to observed sitautions. In this case, either the five-month period where direct labor per month fell to 4% of its previous average did not constitute a production break, or the production break did not result in a significant loss of learning.

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